Mathématiques discrètes et applications
Discrete Mathematics and Applications

Problem Session 3

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Ex-1: Quantified statements.

(a) $\exists x(C(x) \wedge D(x) \wedge F(x))$

(b) $\forall x(C(x) \vee D(x) \vee F(x))$

(c) $\exists x(C(x) \wedge F(x) \wedge \neg D(x))$

(d) $\neg \exists x(C(x) \wedge D(x) \wedge F(x)) \equiv \forall x(\neg C(x) \vee \neg D(x) \vee \neg F(x))$

(e) $\exists x C(x) \wedge \exists y D(y) \wedge \exists z F(z) \equiv \exists x \exists y \exists z(C(x) \wedge D(y) \wedge F(z))$

We can also write the proposition as:
$\exists x C(x) \wedge \exists x D(x) \wedge \exists x F(x)$. Note that x can refer to a different person in each term.

Ex-2: Quantified statements.

(a) $\forall x F(x, Nancie)$

(b) $\forall y F(Peter, y)$

(c) $\forall x \exists y F(x, y)$

(d) $\neg \exists x \forall y F(x, y) \equiv \forall x \exists y \neg F(x, y)$

(e) $\forall y \exists x F(x, y)$

(f)
$$\begin{aligned} \neg \exists x (F(x, Ouny) \wedge F(x, Leyla)) &\equiv \forall x \neg (F(x, Ouny) \wedge F(x, Leyla)) \\ &\equiv \forall x (\neg F(x, Ouny) \vee \neg F(x, Leyla)) \end{aligned}$$

Ex-2: Quantified statements.

(g) $\exists y_1 \exists y_2 (y_1 \neq y_2 \land F(Essil, y_1) \land F(Essil, y_2) \land \forall y (F(Essil, y) \rightarrow (y = y_1 \lor y = y_2)))$

(h) $\neg \exists y F(Ehlena, y) \equiv \forall y \neg F(Ehlena, y)$

(i) $\exists ! y \forall x F(x, y) \equiv \exists y (\forall x F(x, y) \land \forall z (\forall w F(w, z) \rightarrow y = z))$

(j) $\forall x \neg F(x, x)$

(k) $\exists x \exists y (x \neq y \land F(x, y) \land \forall z (F(x, z) \rightarrow z = y))$

Ex-3: Definition of a limit

· We have: $\lim_{x\to a} f(x) = L \iff$

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\ |x - a| < \delta \to |f(x) - L| < \varepsilon$$

· If we name the proposition on the right P, then $\lim_{x\to a} f(x) \neq L$ can be interpreted as $\neg P$.
· Thus we have $\neg P$ as:

$$\begin{aligned} \neg P &\equiv \neg(\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\ |x - a| < \delta \to |f(x) - L| < \varepsilon) \\ &\equiv \exists \varepsilon > 0\ \forall \delta > 0\ \exists x\ (|x - a| < \delta \wedge |f(x) - L| \geq \varepsilon) \end{aligned}$$

· Note that the restricted domains (e.g., $\varepsilon > 0$ or $\delta > 0$) on which the quantifiers apply are not changed when the negation is done. However, we negate the conditional statement being quantified.

Ex-4: Rules of Inference

Consider the following statements:

p : It's sunny this afternoon.

q : It's colder than yesterday.

r : We will go swimming.

s : We will take a canoe trip.

t : We will be home by sunset.

Premises:

1. $\neg p \wedge q$
2. $r \rightarrow p$
3. $\neg r \rightarrow s$
4. $s \rightarrow t$

Conclusion: t

Ex-4: Rules of Inference

We construct our argument with the premises and rules of inference to arrive at the conclusion.

Step	Reason
$\neg r \rightarrow s$	Premise 3
$s \rightarrow t$	Premise 4
$\neg r \rightarrow t$	Transitivity/Hypothetical syllogism
$\neg p \wedge q$	Premise 1
$\neg p$	Specialization/Simplification
$r \rightarrow p$	Premise 2
$\neg r$	Modus Tollens
t	Modus Ponens (see $\neg r \rightarrow t$ above)

Ex-5: Mathematical Induction

Let $P(n)$ be the statement: $\forall n \in \mathbb{N},\ n(n+1)(n+2)$ is divisible by 6.

Basis Step: $n = 0$

$P(0)$: $0(0+1)(0+2) = 0$, which is divisible by 6. So $P(0)$ is true.

Inductive Step: We assume $P(k)$ is true for an arbitrary $k \geq 0$, which is our inductive hypothesis (I.H.):

$P(k)$: $k(k+1)(k+2)$ is divisible by 6.

Assuming the I.H., we show that $P(k+1)$ is true.

· **NB**: The truth of $P(k+1)$ must follow from the I.H., i.e., from the truth of $P(k)$. You do not show that $P(k+1)$ is true on its own.

Ex-5: Mathematical Induction

$$
\begin{aligned}
P(k+1): \ (k+1)(k+2)(k+3) &= k(k+1)(k+2) + 3(k+1)(k+2) \\
&\overset{\text{I.H}}{=} 6M + 3(k+1)(k+2), \ M \in \mathbb{Z} \\
&\overset{(*)}{=} 6M + 3 \cdot 2 \cdot N, \ N \in \mathbb{Z} \\
&= 6M + 6N \\
&= 6(M+N)
\end{aligned}
$$

For the equality $(*)$, we have used that $(k+1)(k+2)$ is even, so $(k+1)(k+2) = 2N$ for some $N \in \mathbb{Z}$.

$6(M+N)$ is divisible by 6. Therefore $P(k) \to P(k+1)$, and by the principle of mathematical induction, $\forall n \in \mathbb{N}$, $P(n)$ holds.

Ex-6: Proof by Contradiction

· To prove a proposition P by contradiction, we show that $\neg P \to \mathbf{F}$. In other words, by assuming P is false, we arrive at a contradiction.
· This indirectly proves P because $P \equiv (\neg P \to \mathbf{F})$.
· This proof method also applies if P is a conditional statements/proposition of the form $A \to B$.

· So to prove by contradiction that: $\forall a \in \mathbb{Z}, a^2 \text{ even } \to a \text{ even}$, we show that the hypothesis $\neg(\forall a \in \mathbb{Z},\ a^2 \text{ even } \to a \text{ even})$ leads to a contradiction.

· $\neg(\forall a \in \mathbb{Z},\ a^2 \text{ even} \to a \text{ even}) \equiv \exists a \in \mathbb{Z} \mid (a^2 \text{ even } \wedge a \text{ odd})$.

$$\begin{aligned} a \text{ odd } &\implies \exists c \in \mathbb{Z} \text{ such that } a = 2c + 1 \\ &\implies a^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1 \text{ is odd} \end{aligned}$$

· We have both a^2 even and a^2 odd, hence a contradiction. So P is true.

Ex-7: Proof by Contraposition.

· To prove that $P \to Q$ by contraposition, we show that $\neg Q \to \neg P$. This proof method is valid because a conditional statement is logically equivalent to its contrapositive.

Consider (for some arbitrary, but fixed a, b, and n) the following statements: $p: \; n = ab$, $q: \; a \leq \sqrt{n}$, and $r: \; b \leq \sqrt{n}$.

We want to prove that: $p \to (q \vee r)$.

By contraposition, it suffices to show that $\neg(q \vee r) \to \neg p$.

$$
\begin{aligned}
\neg(q \vee r) &\equiv \neg q \wedge \neg r \\
&= (a > \sqrt{n}) \wedge (b > \sqrt{n}) \\
&\implies ab > \sqrt{n}\sqrt{n} = n \\
&\implies ab \neq n
\end{aligned}
$$

i.e $\neg(q \vee r) \to \neg p$. $\square$

Ex-8: Mathematical Induction

Let $P(n)$ be the statement:

$$\forall n \geq 1,\ \forall (p_1, \ldots, p_n) \in \{0,1\}^n : \neg(p_1 \wedge \ldots \wedge p_n) = \neg p_1 \vee \ldots \vee \neg p_n$$

Basis Step: For $n = 1$, $\neg p_1 = \neg p_1$, so $P(1)$ is true (there is nothing to do).
For $n = 2$, $\neg(p_1 \wedge p_2) = \neg p_1 \vee \neg p_2$ (shown using truth tables). So also $P(2)$ is true.

Inductive Step: We assume $P(k)$ is true for an arbitrary $k \geq 2$ (Inductive Hypothesis):

$$P(k) : \forall (p_1, \ldots, p_k) \in \{0,1\}^k :\ \neg(p_1 \wedge \ldots \wedge p_k) = \neg p_1 \vee \ldots \vee \neg p_k$$

and show that if $P(k)$ is true, then $P(k+1)$ is true.
$P(k+1)$ is the following statement:

$$\forall (p_1, \ldots, p_{k+1}) \in \{0,1\}^{k+1} :$$

$$\neg(p_1 \wedge \ldots \wedge p_k \wedge p_{k+1}) = \neg p_1 \vee \ldots \vee \neg p_k \vee \neg p_{k+1}$$

Ex-8: Mathematical Induction

$$\begin{aligned}
\neg(p_1 \wedge \ldots \wedge p_k \wedge p_{k+1}) &= \neg((p_1 \wedge p_2 \wedge \ldots \wedge p_k) \wedge p_{k+1}) \\
&= \neg(p_1 \wedge \ldots \wedge p_k) \vee \neg p_{k+1} \qquad \textbf{case } n = 2 \\
&\overset{\text{I.H}}{=} (\neg p_1 \vee \ldots \vee \neg p_k) \vee \neg p_{k+1} \\
&= \neg p_1 \vee \ldots \vee \neg p_k \vee \neg p_{k+1}
\end{aligned}$$

Therefore $P(k) \rightarrow P(k+1)$ and by the principle of mathematical induction, $\forall n \in \mathbb{N}$, $P(n)$ holds.

TODO at home: Similar proof for the second law.

Ex-9: Proving conjectures by mathematical induction.

By computing the sum for small values of n, we can conjecture that the following statement $P(n)$ holds for all n:

$$P(n) : \sum_{i=1}^{n} \frac{1}{2^i} = \frac{2^n - 1}{2^n}$$

Let us prove our conjecture by mathematical induction.

Basis Step: $n = 1$

$P(1)$: $\frac{1}{2^1} = \frac{2^1-1}{2^1} = \frac{1}{2}$. So $P(1)$ is true.

Inductive Step: We assume $P(k)$ is true for an arbitrary $k \geq 1$ (I.H.)

$P(k) : \sum_{i=1}^{k} \frac{1}{2^i} = \frac{2^k-1}{2^k}$

Assuming the I.H., we show that $P(k+1)$ is true.

Ex-9: Proving conjectures by mathematical induction.

$$
\begin{aligned}
P(k+1): \sum_{i=1}^{k+1} \frac{1}{2^i} &= \sum_{i=1}^{k} \frac{1}{2^i} + \frac{1}{2^{k+1}} \\
&\overset{\text{I.H}}{=} \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} \\
&= \frac{2(2^k - 1) + 1}{2^{k+1}} \\
&= \frac{2 \cdot 2^k - 2 + 1}{2} \\
&= \frac{2^{k+1} - 1}{2^{k+1}}
\end{aligned}
$$

Therefore $P(k) \to P(k+1)$, and by the principle of mathematical induction, $\forall n \in \mathbb{N}$, $P(n)$ holds.

End

For more exercises and notes, see Rosen 7th Edition Chap. 1.4-1.8 and Chap. 5.1.