University of Neuchâtel
Discrete Mathematics and Applications - Fall 2025
Problems - 4

1. Let A, B, C be three finite sets.
   (a) Show using set identities that $(A - B) - (B - C) = A - B$. You may assume that A, B, C are subsets of some universal set and use that $A - B = A \cap \overline{B}$ and $B - C = B \cap \overline{C}$.
   (b) Show that $|A \cup B| = |A| + |B| - |A \cap B|$ holds.
2. For any three sets A, B, C, show by double inclusion that $(B - A) \cup (C - A) = (B \cup C) - A$
3. For some set X, we denote by $\mathcal{P}(X)$ the power set of X.
   (a) Find the power sets of the following sets:
   $A = \varnothing$; $B = \{\varnothing\}$; $C = \{a, b, c\}$.
   (b) Prove by induction that: if $|X| = n$, then $|\mathcal{P}(X)| = 2^n$.
4. Let $f : B \to C$ and $g : A \to B$ be two functions; $f \circ g : A \to C$. Show that:
   (a) If $f \circ g$ is injective (one-to-one), then g is injective.
   (b) If $f \circ g$ is surjective, then f is surjective.
5. Let $f : \{1, 2, 3\} \to \{a, b, c, d\}$ be a function defined as: $f(1) = a$, $f(2) = c$, $f(3) = d$. Let $g : \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$ be another function defined as: $g(a) = 2$, $g(b) = 1$, $g(c) = 4$, $g(d) = 5$. Find the composite function $g \circ f$. Is $g \circ f$ surjective?
6. Determine whether the following function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is surjective:
   (a) $f(m, n) = 2m - n$
   (b) $f(m, n) = m^2 - n^2$
   (c) $f(m, n) = m + n - 4$
7. Suppose we have a function $f : A \to B$. Let S and T be subsets of B. Show that:
   (a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$
   (b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$
   (c) $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$
8. Show that the set of positive odd integers $\mathbb{O}$ is countable.
9. Let A and B be two disjoint countable sets. Show that $A \cup B$ is countable.
10. Let A and B be sets such that $A \subseteq B$. By admitting the following lemma:

    Lemma 1. *Every subset of* $\mathbb{N}^*$ *is countable.*

    Show that if B is countable, then A is countable.