Mathématiques discrètes et applications
Discrete Mathematics and Applications

Problem Session 5

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Ex-1a

· **Product rule**: If a task T can be subdivided into n subtasks $T_1, T_2, \ldots, T_n$, and T_1 can be done in k_1 ways, T_2 in k_2 ways, etc., then the total number of ways to do task T is $k_1 \times k_2 \times \cdots \times k_n$.

· We can divide the task of creating a license plate into 6 tasks: each task consisting of filling in the different spots, i.e., sequence of 3 uppercase letters or 3 digits.

· There are 26 ways to fill each of the first 3 spots and 10 ways to fill each of the last 3 spots. So the total number of possible license plates is: $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17'576'000$.

Ex-1b

· **Sum rule**: If a task T can be done in n **different ways**, and there are k_1 possibilities to do it in the first way, k_2 possibilities to do it in the second way, etc., then the total number of ways to do task T is $k_1 + k_2 + \cdots + k_n$.

· We can apply the sum rule here by summing up the possibilities for the 2 different ways of creating a variable name. For a variable name with a single lowercase letter, we have 26 possibilities. For a variable name with a single lowercase letter followed by a digit, we have 26×10 possibilities (i.e., product rule).

· So the total number of possible variable names is
$26 + (26 \times 10) = 286$

Ex-2

· We can first apply the sum rule: a string of length 5 can have length 1, 2, 3, 4, or 5. So we can count the number of possible strings of length i for $i = 1, \ldots, 5$ and then sum up the number of possibilities.

· To count the number of strings of a given length, we can apply the product rule: there are $26^1 = 26$ strings of length 1, $26^2 = 26 \cdot 26$ strings of length 2, etc., up to 26^5 strings of length 5.

· So the total number of possible strings is

$$26 + 26^2 + 26^3 + 26^4 + 26^5 = 12'356'630$$

Ex-3

$\cdot$ Consider the following sets: A $= \{$bit strings that begin with 3 0s$\}$ and B $= \{$bit strings that end with 2 0s$\}$. The question requires us to find $|A \cup B|$.

$\cdot$ For a bit string in set A, there is only 1 way to fill the first 3 spots (i.e., with a 0). So $|A| = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$
$\cdot$ Similarly, for a bit string in set B, there is only 1 way to fill the last 2 spots (i.e., with a 0). So $|B| = 2 \cdot 2 \cdots 2 \cdot 1 \cdot 1 = 2^8$

$A \cap B$ represents strings that begin with 3 0s **and** end with 2 0s. This means there is only one possibility for the first 3 and last 2 spots $\implies |A \cap B| = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 2^5$

$|A \cup B| = |A| + |B| - |A \cap B| = 2^7 + 2^8 - 2^5 = 352$ bit strings.

Ex-4

· Num. of integers divisible by i in range $[1, n] = \lfloor \frac{n}{i} \rfloor$

· Consider the following sets:
A: integers divisible by 2 in $[1, 100]$
B: integers divisible by 3 in $[1, 100]$
C: integers divisible by 5 in $[1, 100]$
U: all integers in $[1, 100]$
Let $x =$ num. of integers in $[1, 100]$ not divisible by 2, 3, or 5.
$\Longrightarrow x = |\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C|$
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$$\Longrightarrow |A| = \lfloor \frac{100}{2} \rfloor = 50$$
$$|B| = \lfloor \frac{100}{3} \rfloor = 33$$
$$|C| = \lfloor \frac{100}{5} \rfloor = 20$$

Ex-4

$$|A \cap B| = \lfloor \frac{100}{\mathrm{lcm}(2,3)} \rfloor = \lfloor \frac{100}{6} \rfloor = 16$$

$$|A \cap C| = \lfloor \frac{100}{\mathrm{lcm}(2,5)} \rfloor = \lfloor \frac{100}{10} \rfloor = 10$$

$$|B \cap C| = \lfloor \frac{100}{\mathrm{lcm}(3,5)} \rfloor = \lfloor \frac{100}{15} \rfloor = 6$$

$$|A \cap B \cap C| = \lfloor \frac{100}{\mathrm{lcm}(2,3,5)} \rfloor = \lfloor \frac{100}{30} \rfloor = 3$$

$$\implies |A \cup B \cup C| = 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$$

$$\begin{aligned} \implies x &= |U| - 74 \\ &= 100 - 74 \\ &= 26 \end{aligned}$$

Ex-5

· **Pigeonhole principle:** if N elements are to be put into k boxes ($N > k$), then there is at least one box with at least $\lceil \frac{N}{k} \rceil$ elements.
· $\lceil x \rceil$ represents the smallest integer greater than or equal to x. That is: $\lceil x \rceil = \min\{n \in \mathbb{Z}, n \geq x\}$.

· We can see the 4369 students here as the elements to be put in boxes, and the boxes are the days of year corresponding to the students' birthdays. From the pigeonhole principle, there is at least one day of year (i.e., box) with at least $\lceil \frac{4369}{366} \rceil = \lceil 11.937 \rceil = 12$ student birthdays.

Ex-6

· The smallest integer N that satisfies the inequality: $\lceil \frac{N}{k} \rceil \geq r$ is $k \times (r - 1) + 1$.

a). The pigeonhole principle can also be applied here by considering the 2 ball colours as pigeonholes. In this case, to find the minimum number of balls to be picked, we need to find the minimum N such that: $\lceil \frac{N}{2} \rceil \geq 3$ which is $2 \times (3 - 1) + 1 = 5$ balls.

b). In the worst case, the first 10 picks will all be red balls. So she must pick at least 11 balls to guarantee there is at least 1 green ball. So to guarantee 3 green balls, she must pick 13 balls.

Ex-7

The last two decimal digits of a positive integer correspond to the remainder when divided by 100. Thus, there are 100 possibilities for the last two decimal digits of a positive integer. Therefore, it suffices to consider the first 101 powers of 2: 1, 2, 4, $\ldots$, 2^{100}. By the pigeonhole principle, two of them have the same last two decimal digits.

(One can be a bit more efficient by noting that, except for 1 and 2, all powers of 2 are divisible by 4, but not by 5, so that there are in fact only 20 possibilities for their last two decimal digits.)

Explicitly, one finds that

$$2^{22} = 4'194'304.$$

End

For more exercises and notes, see Rosen, 7th Edition, Chap. 6 and 8.5.